CONNECTED COMPONENTS OF SPACES OF SURFACE GROUP REPRESENTATIONS II

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Abstract. In [HL1], we discussed the connected components of the space of surface group representations for any compact connected semisimple Lie group and any closed compact (orientable or nonorientable) surface. In this sequel, we generalize the results in [HL1] in two directions: we consider general compact connected Lie groups, and we consider all compact surfaces, including the ones with boundaries. We also interpret our results in terms of moduli spaces of flat connections over compact surfaces.

1. introduction

In [G2], W. M. Goldman computed the number of connected components of spaces of surface group representations $\mathrm{Hom}(\pi_1(\Sigma), G)$ for $G = PSL(2,\mathbb{C})$ or $PSL(2,\mathbb{R})$, and Σ is a Riemann surface of genus bigger than 1. In the same paper, he also made a conjecture for the general connected complex semisimple Lie group G that there is a bijection between $\pi_0(\mathrm{Hom}(\pi_1(\Sigma), G)/G)$ and $H^2(\Sigma, \pi_1(G)) \cong \pi_1(G)$, where the G-action on $\mathrm{Hom}(\pi_1(\Sigma), G)$ is induced by the conjugate action of G on itself. This conjecture was proved by J. Li in [Li].

In [HL1], we computed $\pi_0(\mathrm{Hom}(\pi_1(\Sigma), G)/G)$ for any compact connected semisimple Lie group G and any closed (orientable or nonorientable) surface Σ with negative Euler characteristic, except for the connected sum of a torus and a Klein bottle. The first goal of this note is to generalize the results in [HL1] to general compact connected Lie groups. The result for orientable surfaces is known:

Theorem 1. *Let Σ be a connected, closed, compact, orientable surface with genus $\ell > 0$. Let G be a compact connected Lie group, and let $G_{ss} = [G, G]$ be the maximal connected semi-simple subgroup of G. Then there is a bijection*

$$\pi_0(\mathrm{Hom}(\pi_1(\Sigma), G)/G) \to \pi_1(G_{ss}).$$

The group $\pi_1(G_{ss})$ in Theorem 1 is a finite abelian group. As we will explain in Section 5, a proof of Theorem 1 can be extracted from [AB]. In this note, we will give another proof of Theorem 1, and derive the following result for non-orientable surfaces:

Theorem 2. *Let Σ be a closed, compact, nonorientable surface which is homeomorphic to the connected sum of m copies of the real projective plane, where $m \neq 1, 2, 4$. Let G be a compact*

Date: September 21, 2004.
1991 *Mathematics Subject Classification.* 53.
Key words and phrases. surface group representations.

connected Lie group. Then there is a bijection

$$\pi_0(\mathrm{Hom}(\pi_1(\Sigma), G)/G) \to \pi_1(G)/2\pi_1(G)$$

where $2\pi_1(G)$ denote the subgroup $\{k^2 \mid k \in \pi_1(G)\}$ of the abelian group $\pi_1(G)$.

The group $\pi_1(G)/2\pi_1(G)$ in Theorem 2 fits in the following short exact sequence of abelian groups:

$$1 \to \pi_1(G_{ss})/2\pi_1(G_{ss}) \to \pi_1(G)/2\pi_2(G) \to (\mathbb{Z}/2\mathbb{Z})^{\dim H} \to 1$$

where $G_{ss} = [G, G]$ and H is the connected component of the identity of the center of G. In particular, $\pi_1(G)/2\pi_1(G)$ is a finite abelian group.

Theorem 1 and Theorem 2 generalize the case where G is semisimple considered in [HL1] and the case where G_{ss} is simply connected considered in [HL2].

The second goal of this note is to generalize the above results to compact surfaces with boundaries. The same question becomes trivial in this case: $\pi_1(\Sigma)$ is a free group when Σ is a compact surface with nonempty boundary, so $\mathrm{Hom}(\pi_1(\Sigma), G)$ and $\mathrm{Hom}(\pi_1(\Sigma), G)/G$ are connected for any compact connected Lie group G. However, the question becomes interesting when we impose some boundary conditions. Let Σ be a (possibly nonorientable) compact surface with boundary components $B_1, \ldots, B_r$. Let

$$\mathrm{Hom}_{\mathcal{C}_1,\ldots,\mathcal{C}_r}(\pi_1(\Sigma), G) = \{\phi \in \mathrm{Hom}(\pi_1(\Sigma), G) \mid \phi([B_j]) \in \mathcal{C}_j\} \tag{1}$$

where $[B_j] \in \pi_1(\Sigma)$ and $\mathcal{C}_1, \ldots, \mathcal{C}_r$ are conjugacy classes in G, also known as *markings* in this context. Note that (1) reduces to $\mathrm{Hom}(\pi_1(\Sigma), G)$ when $r = 0$. The conjugate action of G on itself induces a G-action on $\mathrm{Hom}_{\mathcal{C}_1,\ldots,\mathcal{C}_r}(\pi_1(\Sigma), G)$. We will compute

$$\pi_0(\mathrm{Hom}_{\mathcal{C}_1,\ldots,\mathcal{C}_r}(\pi_1(\Sigma), G)/G) = \pi_0(\mathrm{Hom}_{\mathcal{C}_1,\ldots,\mathcal{C}_r}(\pi_1(\Sigma), G)).$$

For orientable surfaces, we have

Theorem 3. *Let Σ be a connected, closed, compact, orientable surface with $\ell > 0$ handles and $r > 0$ boundary components. Let G be a compact connected Lie group, and let $G_{ss} = [G, G]$ be the maximal connected semi-simple subgroup of G. Then*

$$\mathrm{Hom}_{\mathcal{C}_1,\ldots,\mathcal{C}_r}(\pi_1(\Sigma), G)/G$$

is nonempty iff $d_1 \cdots d_r \in G_{ss}$ for some (and therefore for all) $(d_1, \ldots, d_r) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_r$. For any conjugacy classes $\mathcal{C}_1, \ldots, \mathcal{C}_r$ of G such that $d_1 \cdots d_r \in G_{ss}$ for some $(d_1, \ldots, d_r) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_r$, there is a bijection

$$\pi_0(\mathrm{Hom}_{\mathcal{C}_1,\ldots,\mathcal{C}_r}(\pi_1(\Sigma), G)/G) \to \pi_1(G_{ss})/J_{\mathcal{C}_1,\ldots,\mathcal{C}_r}$$

where $J_{\mathcal{C}_1,\ldots,\mathcal{C}_r}$ is a subgroup of $\pi_1(G_{ss})$ defined in Section 2.2.

Let Σ and G be as Theorem 3. By Theorem 3, when G is semisimple and simply connnected, for example, $SU(n)\,(n \geq 2)$, $Spin(n)\,(n \geq 3)$, $Sp(n)\,(n \geq 1)$, $\mathrm{Hom}_{\mathcal{C}_1,\ldots,\mathcal{C}_r}(\pi_1(\Sigma), G)/G$ is nonempty and connected for any conjugacy classes $\mathcal{C}_1, \ldots, \mathcal{C}_r$; when G is not semisimple and

G_{ss} is simply connected, for example, $U(n)$ or compact torus, $\mathrm{Hom}_{\mathcal{C}_1,\ldots,\mathcal{C}_r}(\pi_1(\Sigma), G)/G$ is either empty or connected. When $G = SO(n)\,(n \geq 3)$, $\mathrm{Hom}_{\mathcal{C}_1,\ldots,\mathcal{C}_r}(\pi_1(\Sigma), G)/G$ is nonempty and has either one or two connected components. We will see later that for generic conjugacy classes $\mathcal{C}_1, \ldots, \mathcal{C}_r$ in $SO(n)$, $J_{\mathcal{C}_1,\ldots,\mathcal{C}_r}$ is trivial, and in that case $\mathrm{Hom}_{\mathcal{C}_1,\ldots,\mathcal{C}_r}(\pi_1(\Sigma), G)/G$ has two connected components.

For nonorientable surfaces, we have

Theorem 4. *Let Σ be a connected, compact, nonorientable surface which is homeomorphic to the connected sum of a sphere with r holes and m copies of the real projective plane, where $m \neq 1, 2, 4$. Let G be a compact connected Lie group. Then there is a bijection*

$$\pi_0(\mathrm{Hom}_{\mathcal{C}_1,\ldots,\mathcal{C}_r}(\pi_1(\Sigma), G)/G) \to \pi_1(G)/J'_{\mathcal{C}_1,\ldots,\mathcal{C}_r}$$

where $J'_{\mathcal{C}_1,\ldots,\mathcal{C}_r}$ is the subgroup of $\pi_1(G)$ generated by $2\pi_1(G)$ and $J_{\mathcal{C}_1,\ldots,\mathcal{C}_r}$.

Note that $\pi_1(G)/J'_{\mathcal{C}_1,\ldots,\mathcal{C}_r}$ is the quotient group of the finite abelian group $\pi_1(G)/2\pi_1(G)$. Let Σ and G be as in Theorem 4. When G is semisimple and simply connected, for example, $SU(n)\,(n \geq 2)$, $Spin(n)\,(n \geq 3)$, $Sp(n)\,(n \geq 1)$, $\mathrm{Hom}(\pi_1(\Sigma), G)/G$ is nonempty and connected for any conjugacy classes $\mathcal{C}_1, \ldots, \mathcal{C}_r$ in G. When $G = U(n)$, $\pi_1(G) = \mathbb{Z}$ and $J_{\mathcal{C}_1,\ldots,\mathcal{C}_r} \subset \pi_1(G_{ss})$ is trivial, so $\mathrm{Hom}_{\mathcal{C}_1,\ldots,\mathcal{C}_r}(\pi_1(\Sigma), G)/G$ has two connected components. When $G = SO(n)$, we have

$$\pi_1(G)/J'_{\mathcal{C}_1,\ldots,\mathcal{C}_r} = \pi_1(G_{ss})/J_{\mathcal{C}_1,\ldots,\mathcal{C}_r}$$

since $\pi_1(G_{ss}) = \pi_1(G) = \mathbb{Z}/2\mathbb{Z}$ and $2\pi_1(G)$ is trivial, so the answer is the same as in the orientable case.

Our proofs of the above results rely on a result of Alekseev, Malkin, and Meinrenken on group-valued moment maps (Fact 8).

The rest of this paper is organized as follows. In Section 2, we give some preliminaries on the structure of compact Lie groups and introduce some definitions. In Section 3, we derive Theorem 1–4 under the additional assumption that G is simply connected. The general case is treated in Section 4: the results for orientable surfaces (Theorem 1 and Theorem 3) are proved in Section 4.1, and the results for nonorientable surfaces (Theorem 2 and Theorem 4) are proved in Section 4.2. In Section 5, we give a geometric interpretation of the above results to the moduli spaces of flat bundles over compact surfaces.

2. Preliminaries

2.1. **Compact connected Lie groups.** Let G be a compact connected Lie group. Let $G_{ss} = [G, G]$ be its commutator group. Then G_{ss} is the maximal connected semisimple subgroup of G. Let H be the connected component of the identity of the center $Z(G)$ of G. Then H is a compact torus. The map $\phi : H \times G_{ss} \to G = HG_{ss}$ given by $(h, g) \mapsto hg$ is a finite cover which is also a group homomorphism. The kernel of ϕ is isomorphic to $D = H \cap G_{ss} \subset Z(G_{ss})$,

which is a finite abelian group. Note that G_{ss} is a normal subgroup of G, and the quotient

$$G/G_{ss} \cong H/D$$

is a compact torus.

Let $\rho_{ss} : \tilde{G}_{ss} \to G_{ss}$ be the universal covering map which is also a group homomorphism. Then $\tilde{G}_{ss}$ is a compact, connected, simply connected Lie group, and $\mathrm{Ker}(\rho_{ss})$ is a subgroup of $Z(\tilde{G}_{ss})$.

Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H respectively, and let $\exp_H : \mathfrak{h} \to H$ be the exponential map. The map

$$\rho : \tilde{G} = \mathfrak{h} \times \tilde{G}_{ss} \to G$$

given by

$$(X, g) \to \exp_H(X)\rho_{ss}(g)$$

is the universal covering map which is also a group homomorphism. Here we give a group structure for $\mathfrak{h}$ the addition operation in the vector space $\mathfrak{h}$ and 0 as the identity element for the group structure. Notice that $\tilde{G}$ is not compact. Let $\pi_1 : H \to H/D$ and $\pi_2 : G_{ss} \to G_{ss}/D$ be natural projections. Then

$$\pi_1 \circ \exp_H : \mathfrak{h} \to H/D \cong G/G_{ss}$$

is the universal covering map which is also a group homomorphism, and

$$\check{\Lambda} = \mathrm{Ker}(\pi_1 \circ \exp_H) \cong \pi_1(G/G_{ss}) \cong \mathbb{Z}^{\dim H}.$$

We have

$$\begin{aligned} \mathrm{Ker}(\rho) &= \{(X, g) \in \mathfrak{h} \times \tilde{G}_{ss} \mid \exp_H(X)\rho_{ss}(g) = e\} \\ &\subset \check{\Lambda} \times \mathrm{Ker}(\pi_2 \circ \rho_{ss}) \subset \check{\Lambda} \times Z(\tilde{G}_{ss}) \subset \mathfrak{h} \times Z(\tilde{G}_{ss}) = Z(\tilde{G}). \end{aligned}$$

The map $(X, g) \mapsto X$ defines a surjective group homomorphism $p : \mathrm{Ker}(\rho) \to \check{\Lambda}$. The kernel of p is $\{0\} \times \mathrm{Ker}(\rho_{ss}) \cong \mathrm{Ker}(\rho_{ss})$. So we have an exact sequence of abelian groups

$$1 \to \mathrm{Ker}(\rho_{ss}) \to \mathrm{Ker}(\rho) \to \check{\Lambda} \to 1, \tag{2}$$

which can be rewritten as

$$1 \to \pi_1(G_{ss}) \to \pi_1(G) \to \pi_1(G/G_{ss}) \to 1. \tag{3}$$

where $\pi_1(G_{ss})$ is a finite abelian group, and

$$\pi_1(G/G_{ss}) = \pi_1(H/D) \cong \mathbb{Z}^{\dim H}.$$

2.2. **Conjugacy classes.** Let G be a connected Lie group with center $Z(G)$. Given $a \in G$, let $G \cdot a$ denote the orbit of a of the conjugate action of G on itself. The left multiplication by $z \in Z(G)$ gives a bijection $G \cdot a \to G \cdot (za)$. Given a conjugacy class $\mathcal{C} = G \cdot a$ and $z \in Z(G)$, let $z\mathcal{C}$ denote the conjugacy class $G \cdot (za)$.

Let K be a subgroup of $Z(G)$. Let K acts on G by left multiplication. This K-action on G commutes with the conjugate action of G and induces a K-action on $Con(G)$, the set of

conjugacy classes of G. Given a conjugacy $\mathcal{C} \in Con(G)$, let $K_{\mathcal{C}}$ denote the stabilizer of $\mathcal{C}$ under the K-action on $Con(G)$. It is straightforward to check that $K_{\mathcal{C}} = K_{z\mathcal{C}}$ for any $\mathcal{C} \in Con(G)$ and $z \in Z(G)$.

Now let G be a compact connected Lie group, and let $\rho : \tilde{G} \to G$ be as in Section 2.1. Then $\tilde{G}$ is a noncompact connected Lie group. The following is obviously true.

Lemma 5. *Let $K = \mathrm{Ker}\rho \subset Z(\tilde{G}) = \mathfrak{h} \times Z(\tilde{G}_{ss})$, and let $\tilde{K} = \mathrm{Ker}(\rho_{ss}) \subset Z(\tilde{G}_{ss})$. Then any conjucay class $\mathcal{D} \in Con(\tilde{G})$ is of the form*

$$\mathcal{D} = \{X\} \times \tilde{\mathcal{C}}$$

for some $X \in \mathfrak{h}$ and $\tilde{\mathcal{C}} \in Con(\tilde{G}_{ss})$, and we have

$$K_{\mathcal{D}} = \{0\} \times \tilde{K}_{\tilde{\mathcal{C}}} \subset \{0\} \times \mathrm{Ker}(\rho_{ss}).$$

Given a conjugacy class $\mathcal{C}$ in G, each connected component of $\rho^{-1}(\mathcal{C})$ is a conjugacy class of $\tilde{G}$. Let $\mathcal{D}$ be a connected component of $\rho^{-1}(\mathcal{C})$. Then $\mathcal{D} \to \mathcal{C}$ is a covering map of finite degree. Actually, the degree of $\mathcal{D} \to \mathcal{C}$ is equal to the cardinality of the subgroup $K_{\mathcal{D}}$ of $\{0\} \times \mathrm{Ker}(\rho_{ss}) \cong \pi_1(G_{ss})$. Let

$$J_{\mathcal{C}_1,\ldots,\mathcal{C}_r} \subset \{0\} \times \mathrm{Ker}(\rho_{ss}) \cong \pi_1(G_{ss})$$

be the subgroup generated by $K_{\mathcal{D}_1}, \ldots, K_{\mathcal{D}_r}$, where $\mathcal{D}_j$ is a connected component of $\rho^{-1}(\mathcal{C}_j)$. Note that the definition of $J_{\mathcal{C}_1,\ldots,\mathcal{C}_r}$ does not depend on the choices of $\mathcal{D}_1, \ldots, \mathcal{D}_r$ because any connected component of $\rho^{-1}(\mathcal{C}_j)$ is of the form $z\mathcal{D}_j$ for some $z \in Z(\tilde{G})$ and $K_{z\mathcal{D}_j} = K_{\mathcal{D}_j}$.

Example 6. *$G = SO(3) = G_{ss}$, $\tilde{G} = Spin(3) = \tilde{G}_{ss}$,and $\rho_{ss} = \rho : \tilde{G} \to G$. We view $Spin(3)$ as a subset of the real Clifford algebra C_3. Choose a maximal torus T of $SO(3)$ as follows:*

$$T = \{R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid -\pi \le \theta \le \pi\} \cong U(1)$$

Then

$$\tilde{T} = \rho^{-1}(T) = \{\eta_\phi = \cos\phi - \sin\phi e_1 e_2 \mid -\pi \le \phi \le \pi\} \cong U(1)$$

is a maximal torus of $Spin(3)$. We have $\rho(\eta_\phi) = R_{2\phi}$, so $\rho|_{\tilde{T}} : \tilde{T} \to T$ can be identified with

$$s : U(1) \to U(1), \quad u \mapsto u^2.$$

Let $W = \{\pm 1\}$ be the Weyl group of $SO(3)$ which is also the Weyl group of $Spin(3)$. Then W acts on $SO(3)$ and $Spin(3)$ by $(-1) \cdot R_\theta = R_{-\theta}$ and $(-1) \cdot \eta_\phi = \eta_{-\phi}$, respectively. Let $\mathcal{C}_\theta \in Con(SO(3))$ be the conjugacy class represented by $R_\theta \in T \subset SO(3)$, and let $\mathcal{D}_\phi \in Con(Spin(3))$ be the conjugacy class represented by $\eta_\phi \in \tilde{T} \subset Spin(3)$. Then

$$\begin{aligned} Con(SO(3)) = T/W &= \{\mathcal{C}_\theta \mid 0 \le \theta \le \pi\} \cong [0, \pi] \\ Con(Spin(3)) = \tilde{T}/W &= \{\mathcal{D}_\phi \mid 0 \le \theta \le \pi\} \cong [0, \pi] \end{aligned}$$

$\rho|_{\tilde{T}} : \tilde{T} \to T$ induces a map $\hat{\rho} : \tilde{T}/W \to T/W$ which can be identified with

$$\hat{s} : [0,\pi] \to [0,\pi], \quad \phi \mapsto \begin{cases} 2\phi & 0 \le \phi \le \frac{\pi}{2} \\ 2(\pi-\phi) & \frac{\pi}{2} \le \phi \le \pi. \end{cases}$$

So for $\theta \in [0,\pi]$, we have $\hat{\rho}^{-1}(\mathcal{C}_\theta) = \{\mathcal{D}_{\frac{\theta}{2}}, \mathcal{D}_{\pi-\frac{\theta}{2}}\}$. Note that $\hat{\rho}^{-1}(\mathcal{C}_\theta)$ consists of two conjugacy classes unless $\mathcal{C}_\theta = \mathcal{C}_\pi$, which is represented by

$$R_\pi = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In this case, $\hat{\rho}^{-1}(\mathcal{C}_\pi)$ consists of a single conjugacy class $\mathcal{D}_{\frac{\pi}{2}}$. So for $\theta \in [0,\pi], \theta \neq \pi$, $\rho^{-1}(\mathcal{C}_\theta) = \mathcal{D}_{\frac{\theta}{2}} \cup \mathcal{D}_{\pi-\frac{\theta}{2}} \to \mathcal{C}_\theta$ is the trivial (disconnected) double cover, while $\rho^{-1}(\mathcal{C}_\pi) = \mathcal{D}_{\frac{\pi}{2}} \to \mathcal{C}_\pi$ is a connected nontrivial double cover.

Let $K = \{\pm 1\} = \mathrm{Ker}(\rho_{ss}) = \mathrm{Ker}(\rho)$. Then K acts on $\tilde{T}$ by $(-1)\cdot\eta_\phi = -\eta_\phi = \eta_{\phi-\pi}$. So K acts on $Con(Spin(3))$ by $(-1)\cdot\mathcal{D}_\phi = \mathcal{D}_{\phi-\pi} = \mathcal{D}_{\pi-\phi}$, i.e., $(-1)\cdot\phi = \pi-\phi$ under the identification $Con(Spin(n)) \cong [0,\pi]$. It is clear from the the above explicit description that for $\phi \in [0,\pi]$,

$$K_{D_\phi} = \begin{cases} \{1\}, & \text{if } \phi \neq \frac{\pi}{2}, \\ \{\pm 1\}, & \text{if } \phi = \frac{\pi}{2}. \end{cases}$$

Then for $\mathcal{C}_1,\ldots,\mathcal{C}_r \in Con(SO(3))$,

$$J_{\mathcal{C}_1,\ldots,\mathcal{C}_r} = \begin{cases} \{1\}, & \text{if none of } \mathcal{C}_1,\ldots,\mathcal{C}_r \text{ is equal to } \mathcal{C}_\pi, \\ \{\pm 1\}, & \text{if one of } \mathcal{C}_1,\ldots,\mathcal{C}_r \text{ is equal to } \mathcal{C}_\pi. \end{cases}$$

The analysis in Example 6 can be generalized to show that $J_{\mathcal{C}_1,\ldots,\mathcal{C}_r}$ is trivial for generic conjugacy classes $\mathcal{C}_1,\ldots,\mathcal{C}_r \in Con(SO(n))$, $n \ge 3$.

2.3. Representation varieties.

Definition 7. *Let G be a connected Lie group. Given nonnegative integers ℓ, r such that $(\ell,r) \neq (0,0)$ and conjugacy classes $\mathcal{C}_1,\ldots,\mathcal{C}_r \in Con(G)$, define*

$$\begin{aligned} \Phi^{\mathcal{C}_1,\ldots,\mathcal{C}_r}_{G,\ell,r} : \quad & G^{2\ell} \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_r \longrightarrow G \\ & (a_1,b_1,\ldots,a_\ell,b_\ell,d_1,\ldots,d_r) \mapsto \prod_{i=1}^{\ell}[a_i,b_i]\prod_{j=1}^{r} d_j. \end{aligned}$$

Let $\Phi_{G,0,0} : \{e\} \to G$ be the inclusion of the trivial subgroup.

Given nonnegative integers ℓ, r, conjugacy classes $\mathcal{C}_1, \dots, \mathcal{C}_r \in Con(G)$, and $z \in Z(G)$, define

$$\begin{aligned}
& X_G^{\ell,r,0}(\mathcal{C}_1, \dots, \mathcal{C}_r; z) = \left(\Phi_{G,\ell,r}^{\mathcal{C}_1,\dots,\mathcal{C}_r}\right)^{-1}(z) \\
& X_G^{\ell,r,1}(\mathcal{C}_1, \dots, \mathcal{C}_r; z) \\
= \ & \{(a_1, b_1, \dots, a_\ell, b_\ell, d_1, \dots, d_r, c) \in G^{2\ell} \times \mathcal{C}_1 \times \dots \times \mathcal{C}_r \times G \mid \prod_{i=1}^{\ell}[a_i, b_i] \prod_{j=1}^{r} d_j c^2 = z\} \\
& X_G^{\ell,r,2}(\mathcal{C}_1, \dots, \mathcal{C}_r; z) \\
= \ & \{(a_1, b_1, \dots, a_\ell, b_\ell, d_1, \dots, d_r, c_1, c_2) \in G^{2\ell} \times \mathcal{C}_1 \times \dots \times \mathcal{C}_r \times G^2 \mid \prod_{i=1}^{\ell}[a_i, b_i] \prod_{j=1}^{r} d_j c_1^2 c_2^2 = z\}
\end{aligned}$$

Since G is a Lie group and any conjugacy class of G is a homogeneous space, for any nonnegative integers ℓ, r and $i = 1, 2, 3$

$$G^{2\ell} \times \mathcal{C}_1 \times \dots \times \mathcal{C}_r \times G^i$$

is a manifold (it is a point when $\ell = r = i = 0$). In this topology, $X_G^{\ell,r,i}(\mathcal{C}_1, \dots, \mathcal{C}_r; z)$ is a closed subset. Let $X_G^{\ell,r,i}(\mathcal{C}_1, \dots, \mathcal{C}_r; z)$ be equipped with the induced topology as a closed subset of $X_G^{\ell,r,i}(\mathcal{C}_1, \dots, \mathcal{C}_r; z)$.

Let $\Sigma_0^{\ell,r}$ be the compact, connected, orientable surface with ℓ handles and r boundary components $B_1, \dots, B_r$, where ℓ, r are nonnegative integers. Let $\Sigma_1^{\ell,r}$ be the connected sum of $\Sigma_0^{\ell,r}$ and $\mathbb{RP}^2$, and let $\Sigma_2^{\ell,r}$ be the connected sum of $\Sigma_0^{\ell,r}$ and a Klein bottle. Any compact surface is of the form $\Sigma_i^{\ell,r}$, where ℓ, r are nonnegative integers and $i = 0, 1, 2$. It is orientable iff $i = 0$; it is closed iff $r = 0$. Then $\mathrm{Hom}_{\mathcal{C}_1,\dots,\mathcal{C}_r}(\pi_1(\Sigma_i^{\ell,r}), G)$ can be identified with $X_G^{\ell,r,i}(\mathcal{C}_1, \dots, \mathcal{C}_r; e)$.

3. Simply connected case

In this section, we consider a compact, connected, simply connected Lie group G. In particular, G is semisimple.

Fact 8 ([AMW]). *Let G be a compact, connected, simply connected Lie group. Let ℓ be a positive integer, and let $\mathcal{C}_1, \dots, \mathcal{C}_r$ be any conjugacy classes in G. Then for any $g \in G$, $(\Phi_{G,\ell,r}^{\mathcal{C}_1,\dots,\mathcal{C}_r})^{-1}(g)$ is nonempty and connected.*

Remark 9. *The reason of Fact 8 is that $\{G^{2\ell} \times \mathcal{C}_1 \times \dots \times \mathcal{C}_r, \Phi_{G,\ell,r}^{\mathcal{C}_1,\dots,\mathcal{C}_r}\}$ is a quasi-Hamiltonian system and thus the preimage of the moment map $\Phi_{G,\ell,r}^{\mathcal{C}_1,\dots,\mathcal{C}_r}$ is nonempty and connected under the above assumption of G.*

We will derive the following result from Fact 8.

Proposition 10. *Let G be a compact, connected, simply connected Lie group. Let ℓ be a positive integer, let r be a nonnegative integer, let $\mathcal{C}_1, \dots, \mathcal{C}_r$ be any conjugacy classes in G,*

and let $z \in Z(G)$. Then

$$X_G^{\ell,r,0}(\mathcal{C}_1,\ldots,\mathcal{C}_r;z),\quad X_G^{\ell,r,1}(\mathcal{C}_1,\ldots,\mathcal{C}_r;z)$$

are nonempty and connected;

$$X_G^{\ell,r,2}(\mathcal{C}_1,\ldots,\mathcal{C}_r;z)$$

is nonempty, and is connected if $\ell > 1$.

Proof. By Fact 8,

$$X_G^{\ell,r,0}(\mathcal{C}_1,\ldots,\mathcal{C}_r;z) = \left(\Phi_{G,\ell,r}^{\mathcal{C}_1,\ldots,\mathcal{C}_r}\right)^{-1}(z).$$

is nonempty and connected. In the rest of the proof, we will consider $X_G^{\ell,r,1}(\mathcal{C}_1,\ldots,\mathcal{C}_r;z)$ and $X_G^{\ell,r,2}(\mathcal{C}_1,\ldots,\mathcal{C}_r;z)$.

We fix a maximal torus T in G. For $r=0$, let $s \in T$ be a square root of z. For $r>0$, $T\cap\mathcal{C}_j$ is nonempty for $j=1,\ldots,r$. We fix $t_j \in T\cap\mathcal{C}_j$, and let $s\in T$ be a square root of $(t_1\cdots t_r)^{-1}z$.

Consider maps

$$\begin{array}{rcl} Q_1 : X_G^{\ell,r,1}(\mathcal{C}_1,\ldots,\mathcal{C}_r;z) & \to & G \\ (a_1,b_1,\ldots,a_\ell,b_\ell,d_1,\ldots,d_r,c) & \mapsto & c \\ Q_2 : X_G^{\ell,r,2}(\mathcal{C}_1,\ldots,\mathcal{C}_r;z) & \to & G^2 \\ (a_1,b_1,\ldots,a_\ell,b_\ell,d_1,\ldots,d_r,c_1,c_2) & \mapsto & (c_1,c_2) \end{array}$$

By Fact 8, $Q_1^{-1}(c)$ is nonempty and connected for any $c\in G$, and $Q_2^{-1}(c_1,c_2)$ is nonempty and connected for any $(c_1,c_2)\in G^2$. So $X_G^{\ell,r,1}(\mathcal{C}_1,\ldots,\mathcal{C}_r;z)$ and $X_G^{\ell,r,2}(\mathcal{C}_1,\ldots,\mathcal{C}_r;z)$ are nonempty. We will show that

(1) For any $c_0\in G$ there is a path $\gamma_1:[0,1]\to X_G^{\ell,r,1}(\mathcal{C}_1,\ldots,\mathcal{C}_r;z)$ such that $\gamma_1(0)\in Q_1^{-1}(s)$ and $\gamma_1(1)\in Q_1^{-1}(c_0)$.
(2) If $\ell>1$, then for any $(c_1,c_2)\in G^2$ there is a path $\gamma_2:[0,1]\to X_G^{\ell,r,2}(\mathcal{C}_1,\ldots,\mathcal{C}_r;z)$ such that $\gamma_2(0)\in Q_2^{-1}(s,e)$ and $\gamma_2(1)\in Q_2^{-1}(c_1,c_2)$.

This will complete the proof.

Given $c_0,c_1,c_2\in G$, there exist $g_0,g_1,g_2\in G$ such that $g_i^{-1}c_ig_i\in T$. Let $\mathfrak{g}$ and $\mathfrak{t}$ be the Lie algebras of G and T, respectively. Let $\exp:\mathfrak{g}\to G$ be the exponential map. Then there exist $\xi_0,\xi_1,\xi_2\in\mathfrak{t}$ such that

$$s^{-1}g_0^{-1}c_0g_0 = \exp(\xi_0),\quad s^{-1}g_1^{-1}c_1g_1=\exp(\xi_1),\quad g_2^{-1}c_2g_2=\exp(\xi_2).$$

Let W be the Weyl group of G, and let $w\in W$ be a Coxeter element (cf:[Hu]). The linear map $w:\mathfrak{t}\to\mathfrak{t}$ has no eigenvalue equal to 1, so there exists $\xi_i'\in\mathfrak{t}$ such that $w\cdot\xi_i'-\xi_i'=\xi_i$. Recall that $W=N(T)/T$, where $N(T)$ is the normalizer of T in G, so $w=aT\in N(T)/T$ for some $a\in G$. We have

$$a\exp(t\xi_i')a^{-1}\exp(-t\xi_i')=\exp(t\xi_i)$$

for any $t\in\mathbb{R}$.

The group G is connected, so there exists a path $\tilde{g}_i : [0,1] \to G$ such that $\tilde{g}_i(0) = e$ and $\tilde{g}_i(1) = g_i$, where $i = 0, 1, 2$. Define $\gamma_1 : [0,1] \longrightarrow G^{2\ell} \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_r \times G$ by

$$\gamma_1(t) = (a(t), b(t), e, \dots, e, d_1(t), \dots, d_r(t), c(t)),$$

where

$$\begin{aligned} a(t) &= \tilde{g}_0(t) a \tilde{g}_0(t)^{-1}, \\ b(t) &= \tilde{g}_0(t) \exp(-2t\xi_0') \tilde{g}_0(t)^{-1}, \\ d_j(t) &= \tilde{g}_0(t) t_j \tilde{g}_0(t)^{-1}, \quad j = 1, \dots, r \\ c(t) &= \tilde{g}_0(t) s \exp(t\xi_0) \tilde{g}_0(t)^{-1}. \end{aligned}$$

Then the image of γ_1 lies in $X_G^{\ell,r,1}(\mathcal{C}_1, \dots, \mathcal{C}_r; z)$,

$$\gamma_1(0) = (a, e, e, \dots, e, t_1, \dots, t_r, s) \in Q_1^{-1}(s),$$

and

$$\begin{aligned} \gamma_1(1) &= (g_0 a g_0^{-1}, g_0 \exp(-2\xi_0') g_0^{-1}, e, \dots, e, g_0 t_1 g_0^{-1}, \dots, g_0 t_r g_0^{-1}, \\ &\quad g_0 s \exp(\xi_0) g_0^{-1} = c) \in Q_1^{-1}(c_0). \end{aligned}$$

For $\ell > 1$, define $\gamma_2 : [0,1] \longrightarrow G^{2\ell} \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_r \times G^2$ by

$$\gamma_2(t) = (a_1(t), b_1(t), a_2(t), b_2(t), e, \dots, e, d_1'(t), \dots, d_r'(t), c_1(t), c_2(t)),$$

where

$$\begin{aligned} a_1(t) &= \tilde{g}_2(t) a \tilde{g}_2(t)^{-1}, \\ b_1(t) &= \tilde{g}_2(t) \exp(-2t\xi_2') \tilde{g}_2(t)^{-1}, \\ a_2(t) &= \tilde{g}_1(t) a \tilde{g}_1(t)^{-1}, \\ b_2(t) &= \tilde{g}_1(t) \exp(-2t\xi_1') \tilde{g}_1(t)^{-1}, \\ d_j'(t) &= \tilde{g}_1(t) t_j \tilde{g}_1(t)^{-1}, \quad j = 1, \dots, r, \\ c_1(t) &= \tilde{g}_1(t) s \exp(t\xi_1) \tilde{g}_1(t)^{-1}, \\ c_2(t) &= \tilde{g}_2(t) \exp(t\xi_2) \tilde{g}_2(t)^{-1}. \end{aligned}$$

Then the image of γ_2 lies in $X_G^{\ell,r,2}(\mathcal{C}_1, \dots, \mathcal{C}_r; z)$,

$$\gamma_2(0) = (a, e, a, e, e, \dots, e, t_1, \dots, t_r, s, e) \in Q^{-1}(s, e),$$

and

$$\begin{aligned} \gamma_2(1) &= (g_2 a g_2^{-1}, g_2 \exp(-2\xi_2') g_2^{-1}, g_1 a g_1^{-1}, g_1 \exp(-2\xi_1') g_1^{-1}, e, \dots, e, \\ &\quad g_1 t_1 g_1^{-1}, \dots, g_1 t_r g_1^{-1}, c_1, c_2) \in Q^{-1}(c_1, c_2). \quad \square \end{aligned}$$

Recall that

$$\mathrm{Hom}_{\mathcal{C}_1, \dots, \mathcal{C}_r}(\pi_1(\Sigma_i^{\ell,r}), G) \cong X_G^{\ell,r,i}(\mathcal{C}_1, \dots, \mathcal{C}_r; e)$$

for $i = 0, 1, 2$. So Proposition 10 implies that Theorem 1–4 hold when G is simply connected:

Corollary 11. *Let G be a compact, connected, simply connected Lie group. Let ℓ be a positive integer, let r be a nonnegative integer, let $\mathcal{C}_1,\ldots,\mathcal{C}_r$ be any conjugacy classes in G, and let $z\in Z(G)$. Then*

$$\mathrm{Hom}_{\mathcal{C}_1,\ldots,\mathcal{C}_r}(\pi_1(\Sigma_0^{\ell,r}),G)/G,\quad \mathrm{Hom}_{\mathcal{C}_1,\ldots,\mathcal{C}_r}(\pi_1(\Sigma_1^{\ell,r}),G)/G$$

are nonempty and connected;

$$\mathrm{Hom}_{\mathcal{C}_1,\ldots,\mathcal{C}_r}(\pi_1(\Sigma_2^{\ell,r}),G)/G,$$

is nonempty, and is connected if $\ell>1$.

4. General case

4.1. **Orientable surfaces.** Let G, G_{ss}, H, D be as in Section 2.1, and let $\pi: G\to G/G_{ss} = H/D$ be the projection. Then π descends to a map $\hat{\pi}: Con(G)\to H/D$. Let $\pi_1: H\to H/D$ be the natural projection. We have the following observation.

Lemma 12. *Let G be a compact connected Lie group. Let ℓ be a positive integer. Then*

$$\mathrm{Im}(\Phi_{G,\ell,0}) = G_{ss}.$$

For $r>0$, we have

$$\mathrm{Im}(\Phi_{G,\ell,r}^{\mathcal{C}_1,\ldots,\mathcal{C}_r}) = hG_{ss}$$

where $h\in H$, $\pi_1(h)=\hat{\pi}(\mathcal{C}_1)\cdots\hat{\pi}(\mathcal{C}_r)\in H/D$.

Proof. It is obvious from the definition that $\mathrm{Im}\,(\Phi_{G,\ell,0})\subset G_{ss}$. Conversely, given $g\in G_{ss}$, let $\tilde{g}\in\tilde{G}_{ss}$ be a preimage of g under $\rho_{ss}:\tilde{G}_{ss}\to G_{ss}$. By Fact 8, there exist $\tilde{a},\tilde{b}\in\tilde{G}_{ss}$ such that $[\tilde{a},\tilde{b}]=\tilde{g}$. Let $a=\rho_{ss}(\tilde{a})$ and $b=\rho_{ss}(\tilde{b})$. Then

$$g=[a,b]=\Phi_{G,\ell,0}(a,b,e,\ldots,e)\in\mathrm{Im}\,(\Phi_{G,\ell,0}).$$

So $G_{ss}\subset\mathrm{Im}(\Phi_{G,\ell,0})$.

It follows from the above $r=0$ case that $\mathrm{Im}(\Phi_{G,\ell,r}^{\mathcal{C}_1,\ldots,\mathcal{C}_r})=G_{ss}$ if $\mathcal{C}_j\subset G_{ss}$ for $j=1,\ldots,r$. In general, $\mathcal{C}_j=G\cdot g_j$ for some $g_j\in G$, and $g_j=h_jg'_j$ for some $h_j\in H$ and $g'_j\in G_{ss}$. We have $\mathcal{C}_j=h_j\mathcal{C}'_j$, where $\mathcal{C}'_j=G\cdot g'_j=G_{ss}\cdot g'_j\subset G_{ss}$, and $\pi_1(h_j)=\hat{\pi}(\mathcal{C}_j)$. So

$$\mathrm{Im}(\Phi_{G,\ell,r}^{\mathcal{C}'_1,\ldots,\mathcal{C}'_r})=G_{ss}. \tag{4}$$

From the definition of $\Phi_{G,\ell,r}^{\mathcal{C}_1,\ldots,\mathcal{C}_r}$, one sees that (4) implies

$$\mathrm{Im}(\Phi_{G,\ell,r}^{h_1\mathcal{C}'_1,\ldots,h_r\mathcal{C}'_r})=h_1\cdots h_rG_{ss},$$

or equivalently,

$$\mathrm{Im}(\Phi_{G,\ell,r}^{\mathcal{C}_1,\ldots,\mathcal{C}_r})=hG_{ss},$$

where $h=h_1\cdots h_r\in H$ and $\pi_1(h_1\cdots h_r)=\hat{\pi}(\mathcal{C}_1)\cdots\hat{\pi}(\mathcal{C}_r)$. $\square$.

Let $\mathcal{C}_1, \ldots, \mathcal{C}_r$ be conjugacy classes of G. From the proof of Lemma 12, $\mathcal{C}_j = h_j \mathcal{C}'_j$ for some $h_j \in H$ and some conjugacy class $\mathcal{C}'_j \subset G_{ss}$. Recall that

$$\mathrm{Hom}_{\mathcal{C}_1,\ldots,\mathcal{C}_r}(\pi_1(\Sigma_0^{\ell,r}), G) \cong X_G^{\ell,r,0}(\mathcal{C}_1, \ldots, \mathcal{C}_r; e) = \left(\Phi_{G,\ell,r}^{\mathcal{C}_1,\ldots,\mathcal{C}_r}\right)^{-1}(e).$$

By Lemma 12, $\left(\Phi_{G,\ell,r}^{\mathcal{C}_1,\ldots,\mathcal{C}_r}\right)^{-1}(e)$ is nonempty iff $h_1 \cdots h_r \in D$. Note that if $d_1 \cdots d_r \in G_{ss}$ for some $(d_1, \ldots, d_r) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_r$, then $h_1 \cdots h_r \in D$ and $d_1 \cdots d_r \in G_{ss}$ for *all* $(d_1, \ldots, d_r) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_r$. So we have

Lemma 13. *Let G be a compact connected Lie group, and let ℓ, r be positive integers. Then $\mathrm{Hom}_{\mathcal{C}_1,\ldots,\mathcal{C}_r}(\pi_1(\Sigma_0^{\ell,r}, G))/G$ is nonempty iff $d_1 \cdots d_r \in G_{ss}$ for some (and therefore for all) $(d_1, \ldots, d_r) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_r$.*

Let $\mathcal{C}_1, \ldots, \mathcal{C}_r$ be conjugacy classes in G such that $d_1 \cdots d_r \in G_{ss}$ for some $(d_1, \ldots, d_r) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_r$. From the above discussion, $\mathcal{C}_j = h_j \mathcal{C}'_j$ for some $h_j \in H$, $\mathcal{C}_j \subset G_{ss}$, and $h = h_1 \cdots h_r \in D$. Note that we may replace h_1 and $\mathcal{C}'_1$ by $h^{-1}h_1 \in H$ and $h\mathcal{C}'_1 \subset G_{ss}$, respectively, so we may assume that $h_1 \cdots h_r = e$. The diffeomorphism

$$G^\ell \times \mathcal{C}'_1 \times \cdots \times \mathcal{C}'_r \longrightarrow G^\ell \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_r$$

given by

$$(a_1, b_1, \ldots, a_\ell, b_\ell, d_1, \ldots, d_r) \mapsto (a_1, b_1, \ldots, a_\ell, b_\ell, h_1 d_1, \ldots, h_r d_r)$$

induces an isomorphism of topological spaces

$$X_G^{\ell,r,0}(\mathcal{C}'_1, \ldots, \mathcal{C}'_r; e) \longrightarrow X_G^{\ell,r,0}(\mathcal{C}_1, \ldots, \mathcal{C}_r; e).$$

So

$$\mathrm{Hom}_{\mathcal{C}'_1,\cdots,\mathcal{C}'_r}(\pi_1(\Sigma_0^{\ell,r}), G)/G \cong \mathrm{Hom}_{\mathcal{C}_1,\ldots,\mathcal{C}_r}(\pi_1(\Sigma_0^{\ell,r}), G)/G.$$

From the above discussion, Theorem 3 follows from Lemma 13 and Theorem 14.

Theorem 14. *Let G be a compact connected Lie group, and let $G_{ss} = [G, G]$ be the maximal connected semisimple subgroup of G. Let ℓ, r be positive integers, and let $\mathcal{C}_1, \ldots, \mathcal{C}_r$ be conjugacy classes of G such that $\mathcal{C}_j \subset G_{ss}$ for $j = 1, \ldots, r$. Then there is a bijection*

$$\pi_0(\mathrm{Hom}_{\mathcal{C}_1,\ldots,\mathcal{C}_r}(\pi_1(\Sigma_0^{\ell,r}), G)) \longrightarrow \pi_1(G_{ss})/J_{\mathcal{C}_1,\ldots,\mathcal{C}_r}$$

where $J_{\mathcal{C}_1,\ldots,\mathcal{C}_r}$ is defined as in Section 2.2.

Proof. Let $\rho : \tilde{G} \to G$ and $\rho_{ss} : \tilde{G}_{ss} \to G_{ss}$ be universal coverings, as before. Let $\mathcal{D}_j$ be a connected component of $\rho^{-1}(\mathcal{C}_j)$. Recall from Section 2.2 that $\mathcal{D}_j$ is a conjugacy class of $\tilde{G} = \mathfrak{h} \times \tilde{G}_{ss}$, and $\rho_j : \mathcal{D}_j \to \mathcal{C}_j$ is a finite cover of degree $|K_{\mathcal{D}_j}|$, where $K_{\mathcal{D}_j}$ is a subgroup of $\{0\} \times \mathrm{Ker}(\rho_{ss}) \cong \pi_1(G_{ss})$. $J_{\mathcal{C}_1,\ldots,\mathcal{C}_r}$ is the subgroup of $\{0\} \times \mathrm{Ker}(\rho_{ss})$ generated by $K_{\mathcal{D}_1}, \cdots, K_{\mathcal{D}_r}$. In the rest of this proof, we will identify $\mathrm{Ker}(\rho_{ss})$ with $\{0\} \times \mathrm{Ker}(\rho_{ss})$. Let

$$P = \rho^{2\ell} \times \rho_1 \times \cdots \times \rho_r : \tilde{G}^{2\ell} \times \mathcal{D}_1 \times \cdots \times \mathcal{D}_r \to G^{2\ell} \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_r.$$

Define

$$o : \mathrm{Hom}_{\mathcal{C}_1,\ldots,\mathcal{C}_r}(\pi_1(\Sigma_0^{\ell,r}), G) \cong X_G^{\ell,r,0}(\mathcal{C}_1,\ldots,\mathcal{C}_r; e) \to \mathrm{Ker}(\rho_{ss})/J_{\mathcal{C}_1,\ldots,\mathcal{C}_r}$$

by

$$(a_1,b_1,\ldots,a_\ell,b_\ell,d_1,\ldots,d_r) \mapsto \left[\prod_{i=1}^{\ell}[\tilde{a}_i,\tilde{b}_i]\prod_{j=1}^{r}\tilde{d}_j\right] \tag{5}$$

where

$$(\tilde{a}_1,\tilde{b}_1,\ldots,\tilde{a}_\ell,\tilde{b}_\ell,\tilde{d}_1,\ldots,\tilde{d}_r) \in P^{-1}(a_1,b_1,\ldots,a_\ell,b_\ell,d_1,\ldots,d_r),$$

and $[k]$ denotes the image of $k \in \mathrm{Ker}(\rho_{ss})$ under the natural projection $\mathrm{Ker}(\rho_{ss}) \to \mathrm{Ker}(\rho_{ss})/J_{\mathcal{C}_1,\ldots,\mathcal{C}_r}$. Note that the definition (5) does not depend on the choice of $(\tilde{a}_1,\tilde{b}_1,\ldots,\tilde{a}_\ell,\tilde{b}_\ell,\tilde{d}_1,\ldots,\tilde{d}_r)$ because $\mathrm{Ker}\rho \subset Z(\tilde{G})$ and

$$\rho_j^{-1}(d_j) = \{k\tilde{d}_j \mid k \in K_{\mathcal{D}_j}\}.$$

The map o descends to a continuous map

$$\bar{o} : \mathrm{Hom}_{\mathcal{C}_1,\ldots,\mathcal{C}_r}(\pi_1(\Sigma_0^{\ell,r}), G)/G \to \mathrm{Ker}(\rho_{ss})/J_{\mathcal{C}_1,\ldots,\mathcal{C}_r}.$$

It is easy to check that P restricts to a surjective map

$$X_{\tilde{G}}^{\ell,r,0}(\mathcal{D}_1,\ldots,\mathcal{D}_r;k) \cong \mathfrak{h}^{2\ell} \times X_{\tilde{G}_{ss}}^{\ell,r,0}(\mathcal{D}_1,\ldots,\mathcal{D}_r;k) \longrightarrow o^{-1}([k])$$

for any $k \in \mathrm{Ker}(\rho_{ss})$. By Fact 8, $X_{\tilde{G}_{ss}}^{\ell,r,0}(\mathcal{D}_1,\ldots,\mathcal{D}_r;k)$ is nonempty and connected for any $k \in \mathrm{Ker}(\rho_{ss})$. So $o^{-1}([k])$ and $\bar{o}^{-1}([k])$ are nonempty and connected for each $[k] \in K/J_{\mathcal{C}_1,\ldots,\mathcal{C}_r}$. This completes the proof. $\square$.

Finally, Theorem 1 follows immediately from Theorem 14 because $\mathrm{Hom}(\pi_1(\Sigma_0^{\ell,0}), G)/G$ can be identified with $\mathrm{Hom}_{\{e\}}(\pi_1(\Sigma_0^{\ell,1}), G)/G$ and $J_{\{e\}}$ is trivial.

4.2. **Nonorientable surfaces.** Let G, G_{ss}, H, D be defined as in Section 2.1, and let $\mathcal{C}_1,\ldots,\mathcal{C}_r$ be $r > 0$ conjugacy classes of G. We have seen in Section 4.1 that $\mathcal{C}_j = h_j\mathcal{C}_j'$ for some $h_j \in H$ and some conjugacy class $\mathcal{C}_j' \subset G_{ss}$. Let $s \in H$ be a square root of $h_1\cdots h_r$. The diffeomorphism

$$\begin{array}{ccc} G^{2\ell}\times\mathcal{C}_1\times\cdots\times\mathcal{C}_r\times G & \longrightarrow & G^{2\ell}\times\mathcal{C}_1'\times\cdots\times\mathcal{C}_r'\times G \\ (a_1,b_1,\ldots,a_\ell,b_\ell,d_1,\ldots,d_r,c) & \mapsto & (a_1,b_1,\ldots,a_\ell,b_\ell,h_1^{-1}d_1,\ldots,h_r^{-1}d_r,sc) \end{array}$$

induces a homeomorphism of (topological) subspaces

$$X_G^{\ell,r,1}(\mathcal{C}_1,\ldots,\mathcal{C}_r;e) \to X_G^{\ell,r,1}(\mathcal{C}_1',\ldots,\mathcal{C}_r';e).$$

Similarly, the diffeomorphism

$$\begin{array}{ccc} G^{2\ell}\times\mathcal{C}_1\times\cdots\times\mathcal{C}_r\times G^2 & \longrightarrow & G^{2\ell}\times\mathcal{C}_1'\times\ldots\times\mathcal{C}_r'\times G^2 \\ (a_1,b_1,\ldots,a_\ell,b_\ell,d_1,\ldots,d_r,c_1,c_2) & \mapsto & (a_1,b_1,\ldots,a_\ell,b_\ell,h_1^{-1}d_1,\ldots,h_r^{-1}d_r,c_1,sc_2) \end{array}$$

induces a homeomorphism

$$X_G^{\ell,r,2}(\mathcal{C}_1,\ldots,\mathcal{C}_r;e) \to X_G^{\ell,r,2}(\mathcal{C}_1',\ldots,\mathcal{C}_r';e).$$

So we have
$$\mathrm{Hom}_{\mathcal{C}_1,\ldots,\mathcal{C}_r}(\pi_1(\Sigma_i^{\ell,r}),G) \cong \mathrm{Hom}_{\mathcal{C}'_1,\cdots,\mathcal{C}'_r}(\pi_1(\Sigma_i^{\ell,r}),G)$$
for $\ell, r > 0$ and $i = 1, 2$. Therefore, Theorem 4 follows from Theorem 15.

Theorem 15. *Let $\rho : \tilde{G} \to G$ and $K = \mathrm{Ker}\rho \cong \pi_1(G)$ be defined as in Section 2.1. Let r be a positive integer, and let $\mathcal{C}_1, \ldots, \mathcal{C}_r$ be conjugacy classes of G such that $\mathcal{C}_j \subset G_{ss}$ for $j = 1, \ldots, r$. Then for $\ell \geq i$, where $i = 1, 2$, there is a bijection*
$$\pi_0(\mathrm{Hom}_{\mathcal{C}_1,\ldots,\mathcal{C}_r}(\pi_1(\Sigma_i^{\ell,r}),G)) \cong K/J'_{\mathcal{C}_1,\ldots,\mathcal{C}_r}$$
where $J'_{\mathcal{C}_1,\ldots,\mathcal{C}_r}$ is the subgroup of K generated by $J_{\mathcal{C}_1,\ldots,\mathcal{C}_r}$ and $2K = \{k^2 \mid k \in K\}$.

Remark 16. *Notice that there is no condition on the conjugacy class for the moduli space to be nonempty.*

Proof. Let $\mathcal{D}_j$ be a connected component of $\mathcal{C}_j$. Since $\mathcal{C}_j \subset G_{ss}$, we may choose $\mathcal{D}_j$ such that $\mathcal{D}_j \subset \{0\} \times \tilde{G}_{ss} \subset \mathfrak{h} \times \tilde{G}_{ss} = \tilde{G}$. Recall that $\rho_j : \mathcal{D}_j \to \mathcal{C}_j$ is a finite cover with degree $|K_{\mathcal{D}_j}|$, and $J_{\mathcal{C}_1,\ldots,\mathcal{C}_r}$ is the subgroup of K generated by $K_{\mathcal{D}_1}, \cdots, K_{\mathcal{D}_r}$. Set
$$P_1 = \rho^{2\ell} \times \rho_1 \times \cdots \rho_r \times \rho : \quad \tilde{G}^{2\ell} \times \mathcal{D}_1 \times \cdots \times \mathcal{D}_r \times \tilde{G} \to G^{2\ell} \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_r \times G$$
$$P_2 = \rho^{2\ell} \times \rho_1 \times \cdots \rho_r \times \rho^2 : \quad \tilde{G}^{2\ell} \times \mathcal{D}_1 \times \cdots \times \mathcal{D}_r \times \tilde{G}^2 \to G^{2\ell} \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_r \times G^2$$
For $i = 1, 2$ we define
$$o : X_G^{\ell,r,i}(\mathcal{C}_1, \ldots, \mathcal{C}_r; e) \to K/J'_{\mathcal{C}_1,\ldots,\mathcal{C}_r}$$
as follows. For $i = 1$, o is given by
$$(a_1, b_1, \ldots, a_\ell, b_\ell, d_1, \ldots, d_r, c) \mapsto \left[\prod_{i=1}^{\ell}[\tilde{a}_i, \tilde{b}_i] \prod_{j=1}^{r} \tilde{d}_j \tilde{c}^2\right] \tag{6}$$
where
$$(\tilde{a}_1\tilde{b}_1, \ldots, \tilde{a}_\ell, \tilde{b}_\ell, \tilde{d}_1, \ldots, \tilde{d}_r, \tilde{c}) \in P_1^{-1}(a_1, b_1, \ldots, a_\ell, b_\ell, d_1, \ldots, d_r, c),$$
and $[k]$ denotes the image of k under the natural projection $K \to K/J'_{\mathcal{C}_1,\ldots,\mathcal{C}_r}$. For $i = 2$, o is given by
$$(a_1, b_1, \ldots, a_\ell, b_\ell, d_1, \ldots, d_r, c_1, c_2) \mapsto \left[\prod_{i=1}^{\ell}[\tilde{a}_i, \tilde{b}_i] \prod_{j=1}^{r} \tilde{d}_j \tilde{c}_1^2 \tilde{c}_2^2\right] \tag{7}$$
where
$$(\tilde{a}_1\tilde{b}_1, \ldots, \tilde{a}_\ell, \tilde{b}_\ell, \tilde{d}_1, \ldots, \tilde{d}_r, \tilde{c}_1, \tilde{c}_2) \in P_2^{-1}(a_1, b_1, \ldots, a_\ell, b_\ell, d_1, \ldots, d_r, c_1, c_2),$$
Note that the definitions (6), (7) do not depend on the choices of
$$(\tilde{a}_1, \tilde{b}_1, \ldots, \tilde{a}_\ell, \tilde{b}_\ell, \tilde{d}_1, \ldots, \tilde{d}_r, \tilde{c}), \quad (\tilde{a}_1, \tilde{b}_1, \ldots, \tilde{a}_\ell, \tilde{b}_\ell, \tilde{d}_1, \ldots, \tilde{d}_r, \tilde{c}_1, \tilde{c}_2)$$
respectively. The map o descends to a continuous map
$$\bar{o} : \mathrm{Hom}_{\mathcal{C}_1,\ldots,\mathcal{C}_r}(\pi_1(\Sigma_i^{\ell,r}, G)/G \to K/J'_{\mathcal{C}_1,\ldots,\mathcal{C}_r}$$
where $i = 1, 2$.

For $i = 1, 2$, it is easy to check that P_i restricts to a surjective map

$$X^{\ell,r,i}_{\tilde{G}}(\mathcal{D}_1, \dots, \mathcal{D}_r; (X, z)) \cong \mathfrak{h}^{2\ell+i-1} \times X^{\ell,r,i}_{\tilde{G}_{ss}}(\mathcal{D}_1, \dots, \mathcal{D}_r; z) \longrightarrow o^{-1}([(X, z)])$$

for any $(X, z) \in K \subset H \times Z(\tilde{G}_{ss})$. By Fact 8, $X^{\ell,r,i}_{\tilde{G}_{ss}}(\mathcal{D}_1, \dots, \mathcal{D}_r; z)$ is nonempty and connected if $\ell \geq i$. So $o^{-1}([k])$ is nonempty and connected if $\ell \geq i$. This completes the proof. $\square$

Finally, Theorem 2 follows directly from Theorem 15 because $\mathrm{Hom}(\pi_1(\Sigma^{\ell,0}_i), G)/G$ can be identified with $\mathrm{Hom}_{\{e\}}(\pi_1(\Sigma^{\ell,1}_i), G)/G$ and $J_{\{e\}}$ is trivial.

5. Moduli spaces of flat principal G-bundles

Let G be a compact connected Lie group, and let Σ be a compact surface. Recall from Section 2.3 that any compact surface is of the form $\Sigma^{\ell,r}_i$ for some nonnegative integers ℓ, r and $i = 0, 1, 2$.

A principal G-bundle P over a compact surface Σ is trivial iff it admits a cross section. The obstruction classes $O_n(P) \in H^n(\Sigma; \pi_{n-1}(G))$ are obstructions to the existence of a cross section of P. They are topological invariants of P. For a surface, the only obstructions are O_1 and O_2. Here we consider connected Lie groups so the first obstruction class O_1 is trivial and we are left with the second obstruction class O_2 only.

Let $\mathrm{Prin}_G(\Sigma)$ denote the moduli space of topological principal G bundles over Σ. Then $P \mapsto O_2(P)$ defines a bijection

$$O_2 : \mathrm{Prin}_G(\Sigma) \to H^2(\Sigma; \pi_1(G))$$

where

$$H_2(\Sigma^{\ell,r}_i; \pi_1(G)) = \begin{cases} \pi_1(G), & r = i = 0, \\ \pi_1(G)/2\pi_1(G), & r = 0, i = 1, 2, \\ 0 & r > 0. \end{cases}$$

We first consider the case $r > 0$. From the above discussion, every principal G-bundle over $\Sigma^{\ell,r}_i$ is trivial. The space of connections on the trivial bundle $\Sigma^{\ell,r}_i \times G$ can be identified with $\Omega^1(\Sigma^{\ell,r}_i, \mathfrak{g})$, the space of $\mathfrak{g}$-valued 1-forms on $\Sigma^{\ell,r}_i$. Let $\mathrm{Hol}_j : \Omega^1(\Sigma^{\ell,r}_i, \mathfrak{g}) \to G$ be the holonomy around the boundary B_j. Given conjugacy classes $\mathcal{C}_1, \dots, \mathcal{C}_r$ of G, let

$$\mathcal{M}_G(\Sigma^{\ell,r}_i; \mathcal{C}_1, \dots, \mathcal{C}_r) = \frac{\{A \in \Omega^1(\Sigma^{\ell,r}_i, \mathfrak{g}) \mid F_A = 0, \mathrm{Hol}_j(A) \in \mathcal{C}_j \text{ for } j = 1, \dots, r\}}{C^\infty(\Sigma^{\ell,r}_i, G)}$$

be the moduli space of gauge equivalence classes of flat connections on $\Sigma^{\ell,r}_i \times G$ with holonomy around B_j in $\mathcal{C}_j$. With suitable choices of orientation on B_j, we have

$$\mathcal{M}_G(\Sigma^{\ell,r}_i; \mathcal{C}_1, \dots, \mathcal{C}_r) \cong \mathrm{Hom}_{\mathcal{C}_1, \dots, \mathcal{C}_r}(\pi_1(\Sigma^{\ell,r}_i), G)/G.$$

So in Theorem 3 and Theorem 4, we may replace $\mathrm{Hom}_{\mathcal{C}_1, \dots, \mathcal{C}_r}(\pi_1(\Sigma), G)/G$ by $\mathcal{M}_G(\Sigma; \mathcal{C}_1, \dots, \mathcal{C}_r)$.

We next consider the case $r = 0$. A principal G-bundle on $\Sigma_i^{\ell,0}$ may be topologically nontrivial. Let $\mathcal{M}_G(\Sigma_i^{\ell,0})$ be the moduli space of gauge equivalence classes of flat G-bundles on $\Sigma_i^{\ell,0}$. Here a flat G-bundle is a principal G-bundle together with a flat G connection. Then

$$\mathcal{M}_G(\Sigma_i^{\ell,0}) \cong \mathrm{Hom}(\pi_1(\Sigma_i^{\ell,0}), G)/G.$$

Let

$$F : \mathrm{Hom}(\pi_1(\Sigma_i^{\ell,0}), G)/G \to \mathrm{Prin}_G(\Sigma_i^{\ell,0}),$$

be the map which sends a flat G-bundle to its underlying topological principal G-bundle. The discussion in [HL1, Appendix A] shows that the map

$$O_2 \circ F : \mathrm{Hom}(\pi_1(\Sigma_i^{\ell,0}), G)/G \to H^2(\Sigma_i^{\ell,0}; \pi_1(G))$$

coincides with

$$\bar{o} : \mathrm{Hom}_{\{e\}}(\pi_1(\Sigma_i^{\ell,1}), G)/G \cong \mathrm{Hom}(\pi_1(\Sigma_i^{\ell,0}), G)/G \to H^2(\Sigma_i^{\ell,0}; \pi_1(G))$$

defined in the proofs of Theorem 14 and Theorem 15 in Section 4. See [HL1, Appendix A] for various interpretations of the obstruction map.

Given a topological principal G-bundle P over Σ, let $\mathcal{M}(P)$ denote the moduli space of flat G-connections on P. Note that the subgroup $\pi_1(G_{ss})$ of $\pi_1(G)$ consists of torsion elements in $\pi_1(G)$. From the above discussion, our proof of Theorem 14 gives the following statement:

Theorem 17. *Let Σ be a connected, closed, compact, orientable surface with genus $\ell > 0$. Let G be a compact connected Lie group, and let P be a principal G-bundle over Σ. Then $\mathcal{M}(P)$ is nonempty if and only if the obstruction class $O_2(P) \in H^2(\Sigma; \pi_1(G)) \cong \pi_1(G)$ is a torsion element. In this case, $\mathcal{M}(P)$ is connected.*

Similarly, our proof of Theorem 15 gives the following statement:

Theorem 18. *Let Σ be a closed, compact, orientable surface which is homeomorphic to m copies of the real projective plane. Let G be a compact connected Lie group, and let P be a principal G-bundle over Σ. Then $\mathcal{M}(P)$ is nonempty. Moreover, $\mathcal{M}(P)$ is connected if $m \neq 1, 2, 4$.*

Note that Theorem 17 implies Theorem 1, and Theorem 18 implies Theorem 2. We now outline a proof of Theorem 17 extracted from [AB]. Let Σ, G be as in Theorem 17, and let P be a principal G-bundle over Σ. By [AB, Proposition 6.16], P possesses a central Yang-Mills connection. It is proved in [AB, Section 10] that the semi-stable stratum, which is an open and nonempty subset in the space of all G-connections on P, contains a unique connected component of the space of Yang-Mills connections on P. This connected component consists of central Yang-Mills connections on P. It is proved in [AB, Section 12] that central Yang-Mills connections on P achieve the absolute minimum of Yang-Mills functional on P. This absolute minimum is a topological invariant $L(P)$ of P which vanishes iff $O_2(P) \in \pi_1(G)$ is a torsion element. The G-bundle P admits a flat G-connection iff $L(P) = 0$, and in this case, the space

of flat G-connections on P is the space of central Yang-Mills connections on P. This proves Theorem 17.

Acknowledgments: It is a pleasure to thank L. Jeffrey, E. Meinrenken, and E. Xia for their suggestions and help during the preparation of this note.

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